Homework set 2, Phys 785, Spring 2014

Reading: Ch. 11.9-11.10, Ch. 12.1, 12.7

All chapters (Ch.), equations (Eq.) and problems (Pb.) refer to Jackson 3rd edition.

Problem 1

A familiar formula in covariant form. After investigating the E&M properties of a class of materials, a physicist summarized the results into a formula in covariant form,

$$J^\alpha - \frac{1}{c^2}(U_\beta J^\beta)U^\alpha = \frac{\sigma}{c}F^{\alpha\beta}U_\beta,$$

in the standard notation of Jackson, e.g., here U^α is the 4-velocity of the material sample.

(a) Simplify the formula in the rest frame K', where the sample sits still, to find out what physical quantity σ is.

(b) In the lab frame, the sample moves on a track with constant velocity $\mathbf{v}$ in given external fields $(\mathbf{E}, \mathbf{B})$, and a charge density ρ is observed in the sample. Use the formula above to show that the current density $\mathbf{J}$ observed in the lab is given by

$$\mathbf{J} = \sigma\gamma[\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B} - \frac{1}{c^2}\mathbf{v}(\mathbf{v}\cdot\mathbf{E})] + \rho\mathbf{v}.$$

[20 pts]

Problem 2

Constructing Lagrange density for 2D electromagnetism. Let us apply what we've learned so far to construct possible theories for "electromagnetism" in a two-dimensional (2D) world, the xy plane. Remember there is no z coordinate, so Lorentz 4-vectors are now replaced by 3-vectors.

The starting point is the 3-vector $A^\mu = (\phi, A_x, A_y)$, or in compact notation $A^\mu = (\phi, \mathbf{A})$, where $\mathbf{A} = A_x\hat{x} + A_y\hat{y}$ and $\mu = 0, 1, 2$. In this 2D world, the magnetic field is defined as $B = \partial_x A_y - \partial_y A_x$, so it is a *scalar* instead of a vector, while the electric field $\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}/c = E_x\hat{x} + E_y\hat{y}$ is a vector. Here, $\nabla\phi \equiv (\partial_x\phi)\hat{x} + (\partial_y\phi)\hat{y}$. The metric tensor $g^{\mu\nu}$ is a 3×3 diagonal matrix with diagonal elements $(1, -1, -1)$.

(a) Define the field strength tensor the usual way, $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$. Find the explicit expression for $F^{\alpha\beta}$ and $F_{\alpha\beta}$ in terms of E_x, E_y, B. Present your results in matrix form.

(b) For given source $J^\alpha = (c\rho, \mathbf{J})$, construct a Lagrangian density $\mathcal{L}_{2D}$ for 2D electromagnetism. It has to be gauge invariant, and a Lorentz scalar. Hint: use the 3D Maxwell theory as a guide, $\mathcal{L}_{2D}$ may depend on A^α, J^α, $F^{\alpha\beta}$ etc.

(c) A famed gauge field theory in 2D is given by the Chern-Simons Lagrangian density

$$\mathcal{L}_{cs} = \frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - A_\mu J^\mu.$$

Here the Levi-Civita symbol $\epsilon^{\mu\nu\rho}$ is the completely antisymmetric third-rank tensor with $\epsilon^{012} = 1$, and κ is a constant. Derive the equations of motion from the Euler-Lagrange equation

$$\partial_\beta \frac{\partial\mathcal{L}_{cs}}{\partial(\partial_\beta A_\alpha)} = \frac{\partial\mathcal{L}_{cs}}{\partial A_\alpha}.$$

Show that the charge density ρ is proportional to the magnetic field B, and an electric field in x direction can induce a current in the y direction. This is very different from Maxwell theory, and seems weird. But this property can be extremely useful: the Chern-Simons theory is used to describe the fractional quantum Hall effect and topological insulators in condensed matter physics.

Solving this problem requires a good understanding of Ch. 12.7. [20 pts]