L3. Covariance of electrodynamics, [Jackson 11.9–10]

- Confirm eqns of electrodynamics invariant under Lorentz transformation
- Rewrite these eqns using Lorentz scalar/vector/tensor to make covariance explicit
- Figure out the transformation law for sources and fields

(I) charge and current ρ, $\vec{J}$

Experiments confirms that charge of electron (as point particle) is Lorentz scalar

$$e = 1.6 \times 10^{-19}\ C$$

charge conservation law

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{J} = 0 \quad \Rightarrow \quad \boxed{\partial_\alpha J^\alpha = 0}$$

covariant form

valid in any frame

recall $\partial_\alpha = \dfrac{\partial}{\partial x^\alpha} = \left(\dfrac{1}{c}\dfrac{\partial}{\partial t},\ \nabla\right)$

introduce 4-vector (Lorentz vector)

$$J^\alpha = (c\rho,\ \vec{J})$$

$$\therefore \quad \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix}' = \begin{pmatrix} & & & \\ & L & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix}$$

(II). Scalar and Vector potential $\phi, \vec{A}$

$\vec{B} = \nabla \times \vec{A}$ the choice of $(\phi, \vec{A})$ not unique $\leftrightarrow$ different gauge

$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla \varphi$ Lorentz gauge: $\frac{1}{c}\frac{\partial \varphi}{\partial t} + \nabla \cdot \vec{A} = 0$

In vaccum $(\vec{E} = \vec{D}, \vec{B} = \vec{H})$, ϕ and $\vec{A}$ satisfy

$$\begin{cases} \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \frac{4\pi}{c}\vec{J} & (1) \\ \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = \frac{4\pi}{c}\rho & (2) \end{cases}$$

Check: from Maxwell eqn,

$$\nabla \times \vec{H} - \frac{1}{c}\partial_t \vec{D} = \frac{4\pi}{c}\vec{J}$$

$$\stackrel{L}{=} \nabla \times (\nabla \times \vec{A}) - \frac{1}{c}\partial_t\left(-\nabla\phi - \frac{1}{c}\partial_t \vec{A}\right)$$

$$= \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} + \frac{1}{c}\nabla \partial_t \phi + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}$$

$$= \nabla\Big[\underbrace{\frac{1}{c}\partial_t \phi + \nabla \cdot \vec{A}}_{\stackrel{L}{=} 0}\Big] + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A}$$

$\Rightarrow$ (1)

$\nabla \cdot \vec{D} = 4\pi \rho$

$\Rightarrow$ (2)

introduce 4-vector, $A^\alpha = (\phi, \vec{A})$

in Lorentz gauge: $\partial_\alpha A^\alpha = 0$, $\boxed{\Box A^\alpha = \frac{4\pi}{c} J^\alpha} \Longleftrightarrow \begin{cases} ① \\ ② \end{cases}$

recall $\partial_\alpha \partial^\alpha \equiv \Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$.

Covariant form

(III) The field strength tensor

we know how $(\phi, \vec{A})$ transforms, how about $\vec{E}$ and $\vec{B}$?

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla \varphi$$

e.g. $E_x = -\frac{1}{c}\partial_t A_x - \partial_x \varphi = -(\partial^0 A^1 - \partial^1 A^0)$.

$E_y = -\frac{1}{c}\partial_t A_y - \partial_y \phi = -(\partial^0 A^2 - \partial^2 A^0)$.

....

recall:

$$\boxed{\begin{aligned} \partial^\alpha &= \left(\tfrac{1}{c}\partial_t, -\nabla\right) \\ A^\alpha &= (\phi, \vec{A}) \end{aligned}}$$

$$\vec{B} = \nabla \times \vec{A}$$

e.g. $B_x = \partial_y A_z - \partial_z A_y = -(\partial^2 A^3 - \partial^3 A^2)$

This suggests that E_x, E_y, E_z and B_x, B_y, B_z are six components of a 2-nd rank tensor.

Field Strength tensor

$$F^{\alpha\beta} = \partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha}$$

(anti-symmetric, 2-nd rank)   contravariant

Explicitly,

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

The covariant field strength tensor

$$F_{\alpha\beta} = g_{\alpha\gamma}\, g_{\beta\delta}\, F^{\gamma\delta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F^{\alpha\beta} \xrightarrow{\vec{E} \to -\vec{E}} F_{\alpha\beta}$$

B_i: involve 2 spatial index, $g_{\alpha\gamma}$ and $g_{\beta\delta}$ same sign.

E_i: 〃 one time and one spatial index, 〃 opposite sign.

$F_{\alpha\beta}$ is anti-symmetric also.

Define dual tensor

$$f^{\alpha\beta} = \frac{1}{2}\,\varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

$$F^{\alpha\beta} \xrightarrow[\vec{B}\to-\vec{E}]{\vec{E}\to\vec{B}} f^{\alpha\beta},$$

$\vec{E}$ and $\vec{B}$ exchange places, known as duality mapping.

(IV) Maxwell eqns in covariant form (in vacuum)

inhomogeous:

① $\begin{cases} \nabla\cdot\vec{E} = 4\pi\rho \\ \nabla\times\vec{B} - \frac{1}{c}\partial_t\vec{E} = \frac{4\pi}{c}\vec{J} \end{cases}$ $\Longleftrightarrow$ $\boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta}$

4-vector J^β

contraction of ∂_α and $F^{\alpha\beta}$ $\Rightarrow$ 4-vector

Check: $\beta = 0$,

$$\text{Left} = \partial_\alpha F^{\alpha 0} = \partial_i F^{i0} = \nabla\cdot\vec{E}$$

$$\text{right} = \frac{4\pi}{c}\cdot c\rho = 4\pi\rho$$

homogeneous :

$$② \begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} + \frac{1}{c} \partial_t \vec{B} = 0 \end{cases} \quad \longleftrightarrow \quad ① \begin{cases} \nabla \cdot \vec{E} = 4\pi \rho \\ \nabla \times \vec{B} - \frac{1}{c} \partial_t \vec{E} = \frac{4\pi}{c} \vec{J} \end{cases}$$

② is related to ① by duality mapping $\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$ for zero source, so

$$② \Longleftrightarrow \partial_\alpha f^{\alpha\beta} = 0$$

(V). Lorentz eqn

$$\frac{d\vec{p}}{dt} = q (\vec{E} + \vec{v} \times \vec{B})$$

To put it in co-variant form, we need to refresh our memory of relativistic mechanics.

Basic 4-vector $\quad x^\alpha = (ct, \vec{x})$

" $\quad$ scalar $\quad d\tau = ds/c$, the proper time

From these, we define

define 4- velocity

$$U^\alpha = \frac{dx^\alpha}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{d\vec{x}}{d\tau} \right)$$

$$= (c\gamma, \gamma\vec{v})$$

Recall:

$dt = \gamma\, d\tau$

$\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$\beta = v/c$

$\vec{v} = \frac{d\vec{x}}{dt}$

$$U^\alpha = (U^0, \vec{u})$$

i.e. $\begin{cases} U^0 = \gamma c \\ \vec{u} = \gamma\vec{v} \end{cases}$

define 4- momentum

$$p^\alpha = m U^\alpha = m(c\gamma, \gamma\vec{v}),$$

m: rest mass of particle

$$p^\alpha = (p^0; \vec{p}) = \left(\frac{E}{c}, \vec{p}\right)$$

i.e. $\begin{cases} p^0 = \frac{E}{c} = mc\gamma \\ \vec{p} = m\gamma\vec{v} \end{cases}$

what's the meaning of E and $\vec{p}$?

in the non-relativistic limit, $v \ll c$, $\gamma \to 1$, $\vec{p} \to m\vec{v}$

$\therefore \vec{p}$ is momentum

For particle at rest, $v=0$, $E_0 = mc^2$, the rest energy

For particle in motion, $E = mc^2 \cdot \gamma$

$$\simeq mc^2 \left[1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots \right] \quad \text{for } u \ll c$$

$$= E_0 + \frac{1}{2} m u^2 + \cdots$$

$\therefore$ E is the energy.

define kinetic energy $T = E - E_0$

$$\to \frac{1}{2} m u^2 \quad \text{for } u \ll c$$

Construct Lorentz scalar

$$p^\alpha p_\alpha = \left(\frac{E}{c}\right)^2 - |\vec{p}|^2$$

$$= m^2 c^2 \quad \text{(e.g. in rest frame)}$$

$$\Rightarrow E = \sqrt{m^2 c^4 + p^2 c^2}$$

the relativistic energy-momentum dispersion

Special case: photon has $m=0$, $E = pc$

in QM: $E = \hbar\omega$, $\vec{p} = \hbar \vec{k}$, so $\left(\frac{\omega}{c}, \vec{k}\right)$ is 4-vector.

back to Lorentz force eqn $\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$

Consider

$$\frac{d\vec{p}}{d\tau} = \frac{d\vec{p}}{dt} \cdot \frac{dt}{d\tau} = \gamma q \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

$$= \frac{q}{c}\left(U^0 \vec{E} + \vec{u} \times \vec{B}\right) \qquad u^\alpha = (c\gamma, \gamma \vec{v})$$

we need to also consider $\qquad p^\alpha = \left(\frac{E}{c}, \vec{p}\right)$

$$\frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{d\tau} = \frac{1}{c}\gamma \frac{dE}{dt}$$

$$= \frac{1}{c}\gamma q \, \vec{E} \cdot \vec{v}$$

$$= \frac{q}{c} \vec{E} \cdot \vec{u}$$

Combine them into

$$\frac{dp^\alpha}{d\tau} = m \frac{du^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} u_\beta$$

Check: $\alpha = 0$, $\quad F^{0\beta} u_\beta = F^{0i} u_i = -E_i(-u_i) = \vec{E} \cdot \vec{u}$, ✓

$\alpha = 1$, $\quad F^{1\beta} u_\beta = E_x U^0 + \underline{F^{1i} u_i}$

$$\underline{F^{1i} u_i} = -B_z(-u_y) + B_y(-u_z)$$

$$= (\vec{u} \times \vec{B})_x \, ,$$ ✓